An Interpolatory Spectral Element Method Using Curl-Conforming Vector Basis Functions on Tetrahedra

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Introduction

The Nedelec basis functions are commonly used in the finite element solution of electromagnetic field problems. Higher-order Nedelec-type of basis functions are constructed to be either hierarchical or interpolatory. Due to a lack of an explicit expression for interpolation functions through an arbitrary set of nodes on a tetrahedron, equispaced nodes, for which explicit expressions do exist, are commonly used in the finite element formulation. The poor interpolation properties of those functions make the finite element matrices poorly conditioned for higher orders. Here we use the Vandermonde matrix to express the interpolatory vector basis on an arbitrary set of nodes in terms of a hierarchical basis utilizing orthonormal polynomials on a tetrahedron. In an effort to increase efficiency, integration and differentiation operations are developed using matrix-matrix multiplications. Numerical results are given which verify the efficacy of the approach.

Spectral Element Method for Scalar Problems

An interpolatory form of the spectral element method (SEM) is a convenient and popular approach to representing the approximate solution. Since there is no closed form expression for such functions on an arbitrary set of nodes, a hierarchical basis can be used as an intermediary. Because the two bases span the same polynomial space, a change of basis (the Vandermonode matrix) can be used to represent the hierarchical basis as a linear combination of the interpolatory basis, and vice-versa. A suitable choice for the hierarchical basis is a set of orthonormal polynomials on tetrahedra, proposed by Proriol, Kornwinder, Dubiner, and Owens (the PKDO polynomials) [1]. It has the advantage of numerical linear independence at high orders and fast numerical evaluation of both the function and its partial derivatives. For the interpolatory basis, the set of collocation points has a significant effect on the conditioning of the Vandermonde matrix. Well-behaved distributions such as the electrostatic, Fekete, and warp and blend nodes are utilized to improve the conditioning [1].

The interpolatory basis using equispaced points has an explicit closed form expression and is commonly used in the electromagnetics community [2]. The use of this basis has several drawbacks. First, the analytical expression does not lend itself to fast evaluation. Second, the equispaced points are not the optimal set of nodes to use to achieve the best accuracy of the field expansion.

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Proposed Spectral Element Method for Vector Problems

Previous work on a developing a SEM using Nedelec bases was limited to hexagonal elements [3]. By developing a method that allows for any standard element, we achieve a greater flexibility in the type of mesh that can be used. For this paper, we consider the tetrahedral element.

Hierarchical and Interpolatory Bases. For a vector SEM problem, a desirable basis should maintain tangential continuity between elements. A mixed-order curl-conforming basis complete to order P can be constructed by forming the product of the zeroth order edge-basis with complete scalar polynomial factors of order P [2]. If ϕ_{pqr} denotes the PKDO polynomial with indices pqr and $\mathbf{N}_\beta$ the zeroth order basis associated with edge β, then the vector hierarchical basis can be formed as $\mathbf{H}^\beta_{pqr} = \phi_{pqr}\mathbf{N}_\beta$.

It should be noted that such a set is not linearly independent. While a Pth-order element has $(P+1)(P+3)(P+4)/2$ degrees of freedom [2], there are $(P+1)(P+2) \cdot (P+3)/6$ scalar polynomials of order P [1] yielding $(P+1)(P+2)(P+3)$ hierarchical basis functions. The linear dependence of this set of functions will be dealt with numerically later on in this paper.

The interpolatory basis will be identical to those in [2] except it will use an arbitrary set of collocation points, rather than equispaced points. Since the hierarchical and interpolatory bases are both curl-conforming, the former can be written as a linear combination of the latter: $\mathbf{H}^\beta_{pqr}(\mathbf{x}) = \sum_{i=1}^{N} c_i \mathbf{L}_i(\mathbf{x})$, where N is the number of basis functions. As in the scalar case, the hierarchical basis allows for the numerical evaluation, differentiation, and integration of the interpolatory basis.

In order to maintain tangential continuity, vertex nodes have zero degrees of freedom (DoF), edge nodes have one DoF, face nodes have two DoFs, and interior nodes have three DoFs. To form the Vandermonde matrix, we perform a collocation projection at the interpolation nodes:

$$\mathbf{H}^\beta_{pqr}(\mathbf{x}_j) = \phi_{pqr}(\mathbf{x}_j)\mathbf{N}_\beta(\mathbf{x}_j) = \sum_{i=1}^{N} c_i \mathbf{L}_i(\mathbf{x}_j) = \sum_{k=1}^{DoF_j} c_{j_k}\mathbf{N}_{\beta_{j_k}}(\mathbf{x}_j) \tag{1}$$

where j_k is the kth DoF associated with the jth node and β_{j_k} is the edge associated with the j_kth DoF.

Unlike in the scalar case, this projection immediately yields coefficients only for nodes residing on edges. For edge nodes, if $\beta = \beta_{j_1}$ then $c_{j_1} = \phi_{pqr}(\mathbf{x}_j)$; otherwise $c_{j_1} = 0$. For face nodes, if edge β lies on the same face as the node, then (1) yields the overdetermined system

$$\phi_{pqr}(\mathbf{x}_j)\mathbf{N}_\beta(\mathbf{x}_j) = c_{j_1}\mathbf{N}_{\beta_{j_1}}(\mathbf{x}_j) + c_{j_2}\mathbf{N}_{\beta_{j_2}}(\mathbf{x}_j) \tag{2}$$

where the left-hand side is in column space of the matrix so that an exact solution exists; otherwise $c_{j_1} = c_{j_2} = 0$. For interior nodes, we can choose any three out of the six edge basis functions to multiply with the PKDO polynomials. The solution of the 3×3 linear system yields the coefficients.

Due to the linear dependence of the set of hierarchical basis functions, the Vandermonde matrix is not square. This can be remedied by using numerical techniques for solving 'underdetermined' systems that actually have a unique solution. The dependent columns can then be removed yielding a square Vandermonde matrix.

Integration. We use Gaussian quadrature to evaluate the integrals of the form $\int_V \mathbf{L}_i^T \mathbf{A} \mathbf{L}_j dV$ where $\mathbf{A}$ is a 3×3 matrix. When expanded, the integral can be be expressed in terms of the components of the basis functions.

Since the interpolatory basis has no explicit form, we first evaluate the hierarchical basis functions at the quadrature points. This must be done for each component, resulting in the three matrices

$$Q_\xi \left[m\left(ijk\right)\right]\left[n\left(pqr\beta\right)\right] = \hat{\xi} \cdot \mathbf{H}_{pqr}^{\beta}\left[\mathbf{x}_{m(ijk)}\right] = \phi_{pqr}\left[\mathbf{x}_{m(ijk)}\right]\left\{\hat{\xi} \cdot \mathbf{N}_\beta\left[\mathbf{x}_{m(ijk)}\right]\right\} \tag{3}$$

where ξ is one of $\{x, y, z\}$. The matrices for the interpolatory basis evaluated at the quadrature nodes can then be computed as $B_\xi = Q_\xi V^{-1}$. The integral in (3) is entry ij in $\sum_{\xi,\eta\in\{x,y,z\}} B_\xi W_{\xi\eta} B_\eta$ where $W_{\xi\eta}$ is a diagonal matrix containing the weights scaled by $A_{\xi\eta}$. Due to a lack of space, we refer the reader to [1] for the details of the integration nodes and weights.

Differentiation. We can compute similar matrices for the partial derivatives of the hierarchical basis evaluated at the quadrature nodes. Since there are three components and three partial derivatives involved in the curl, there are nine matrices to calculate:

$$\begin{aligned} D_\xi^{\partial\eta}\left[m\left(ijk\right)\right]\left[n\left(pqr\beta\right)\right] &= \frac{\partial}{\partial\eta}\left(\hat{\xi} \cdot \mathbf{H}_{pqr}^{\beta}\right)\Big|_{\mathbf{x}_{m(ijk)}} \\ &= \hat{\xi} \cdot \left(\frac{\partial\phi_{pqr}}{\partial\eta}\mathbf{N}_\beta + \phi_{pqr}\frac{\partial\mathbf{N}_\beta}{\partial\eta}\right)\Big|_{\mathbf{x}_{m(ijk)}} \end{aligned} \tag{4}$$

where ξ, η are one of $\{x, y, z\}$. We can now calculate the matrices of the partial derivatives of the interpolatory basis functions evaluated at the quadrature nodes. This allows calculation of the three components of the curl as $C_\zeta = \left(D_\xi^{\partial\eta} - D_\eta^{\partial\xi}\right) V^{-1}$, where $\zeta\eta\xi$ is a cyclic permutation of xyz.

Eigenvalue Example

We consider the generalized eigenvalue problem for the vector wave equation given by

$$\nabla \times \nabla \times \mathbf{E} - k_0^2 \mathbf{E} = \mathbf{0} \quad \text{and} \quad \hat{\mathbf{n}} \times \mathbf{E}\,|_{bdy} = \mathbf{0} \tag{5}$$

We use the Galerkin method to convert this to a generalized matrix eigenvalue problem

$$\mathbf{SE} = k_0^2 \mathbf{ME} \tag{6}$$

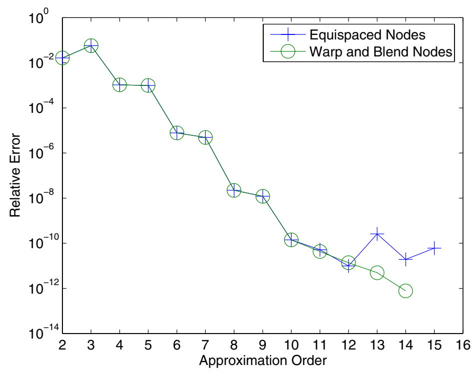
where $\mathbf{S}$ is the stiffness matrix and $\mathbf{M}$ is the mass matrix. The individual entries of the eth element are given by

$$S_{ij}^e = \int_V \frac{1}{|\mathbf{J}|}\left(\nabla \times \mathbf{L}_i\right)^T \mathbf{J}\mathbf{J}^T \left(\nabla \times \mathbf{L}_j\right) dV \tag{7}$$

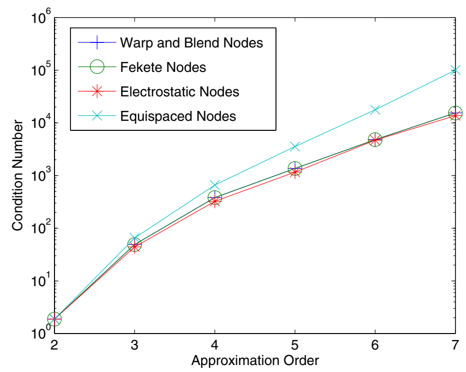
$$M_{ij}^e = \int_V \frac{1}{|\mathbf{J}|}\mathbf{L}_i^T \mathbf{J}^{-T}\mathbf{J}^{-1}\mathbf{L}_j dV \tag{8}$$

where V denotes the volume of the reference tetrahedron and $\mathbf{J}$ is the Jacobian matrix [3].

To validate this method, we calculated the eigenvalues of a single tetrahedron and examined the convergence of the first eigenvalue as the order of the approximation was increased. We also calculated the condition number of $\mathbf{S}-k_0^2\mathbf{M}$ with $k_0^2 = 10.509$ for different sets of nodes.



Error in the first eigenvalue



$\mathrm{cond}\left(\mathbf{S} - k_0^2\mathbf{M}\right)$

Conclusion

The proposed vector SEM performs all numerical operations on a standard reference element so they can be done in pre-processing. Thus, to build each element's mass and stiffness matrices, we need only to perform a constant number of scalar-matrix multiplcations and matrix-matrix additions. In the case of curved elements where the Jacobian is non-constant, we would need to perform the final set of matrix-matrix multiplications at each step.

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